JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 10, 325-329 (1965)

On a Generalization of a Theorem of Lyapunov

HUBERT HALKIN

Bell Telephone Laboratories, Whippany, New Jersey

Submitted by J. P. LaSalle

INTRODUCTION AND SUMMARY

We denote by $\mathscr{B}$ the class of Borel subsets of [0, 1] and by μ the Lebesgue measure. An algebra $\mathscr{A} \subset \mathscr{B}$ such that for every $A \in \mathscr{A}$ there exists a class of sets $\{D_\alpha : \alpha \in [0, 1]\} \subset \mathscr{A}$ with $D_1 = A$, $\mu(D_\alpha) = \alpha\mu(A)$ and $D_{\alpha_1} \subset D_{\alpha_2}$ if $\alpha_1 < \alpha_2$, is called a continuous subalgebra of $\mathscr{B}$. Let Γ be the set of all continuous subalgebras of $\mathscr{B}$. Let X be a finite dimensional Euclidean space. If $\mathscr{A} \subset \mathscr{B}$ let $F(\mathscr{A})$ be the class of Lebesgue integrable functions from [0, 1] into X such that for any $f \in F(\mathscr{A})$ and $p \in X$, we have

$$\{t : p \cdot f(t) > 0, t \in [0, 1]\} \in \mathscr{A}$$

where $p \cdot f(t)$ denotes the scalar product of p and $f(t)$. If $f \in F(\mathscr{B})$ and $\mathscr{A} \subset \mathscr{B}$ let

$$L(f, \mathscr{A}) = \left\{\int_D f d\mu : D \in \mathscr{A}\right\}.$$

In this paper we shall prove the following theorem:

THEOREM I. *If $\mathscr{A} \in \Gamma$ and $f \in F(\mathscr{A})$ then $L(f, \mathscr{A})$ is closed and convex.*

SOME PARTICULAR CASES OF APPLICATIONS FOR THEOREM I

Case 1. $\mathscr{A} = \mathscr{B}$ and f is an integrable function.

Case 2. $\mathscr{A}$ is the class of subsets of [0, 1] whose characteristic functions are continuous at all points of [0, 1] with the exception of at most a countable number and f is an integrable function which is analytic at all points of [0, 1] with the exception of at most a countable number.

Case 3. $\mathscr{A}$ is the class of sets which are the union of a finite number of disjoint intervals of [0, 1] and f is a piecewise constant function.

The application of Theorem I to Case 1 gives immediately the well known Theorem of Lyapunov [1-6]. The application of Theorem I to Case 3 leads to a theorem which is very easy to prove directly. The application of Theorem I to Case 2 leads to some new results in the theory of optimal control.

For instance we can obtain a stronger version of LaSalle's "Bang-Bang" Principle [7]: "If there is an optimal steering function, then there is always a bang-bang steering function that is optimal." In LaSalle's version of this principle a bang-bang steering function is a measurable function taking its values on the vertices of some given hypercube. Applying Case 2 of Theorem I it is easy to prove the same principle where a bang-bang steering function is furthermore restricted to be continuous at every point with the exception of atmost a countable number.

We conjecture that many other results in the theory of optimal control [8-11] could be similarly strengthened.

Proof of Theorem I

Let $\mathscr{A} \in \Gamma$ and $f \in F(\mathscr{A})$. Let $\{D_\alpha : \alpha \in [0, 1]\}$ be a subset of $\mathscr{A}$ with $D_1 = [0, 1]$, $\mu(D_\alpha) = \alpha$ and $D_{\alpha_1} \subset D_{\alpha_2}$ if $\alpha_1 < \alpha_2$. For each $\alpha \in [0, 1]$ let $f_\alpha \in \mathscr{A}$ be defined as follows:

$$\begin{aligned} f_\alpha(t) &= f(t) \qquad \text{if} \qquad t \in D_\alpha \\ &= 0 \qquad\quad\ \text{if} \qquad t \in [0, 1] \sim D_\alpha . \end{aligned}$$

We shall write $L(\alpha)$ for $L(f_\alpha, \mathscr{A})$ and L for $L(1)$. We have then $L(\alpha_1) \subset L(\alpha_2)$ if $\alpha_1 < \alpha_2$. Let

$$K = \bigcup_{\alpha \in [0,1]} \partial \operatorname{co} L(\alpha)$$

where $\partial \operatorname{co} L(\alpha)$ means the boundary of the convex hull of the set $L(\alpha)$. In Lemma I we shall prove that $\operatorname{co} \bar{L} \subset K$. In Lemma II we shall prove that $K \subset L$. It follows then that $\operatorname{co} \bar{L} \subset L$. In other words the set L is closed and convex. This concludes the proof of Theorem I.

Lemma I.

$$\operatorname{co} \bar{L} \subset K .$$

Proof of Lemma I: Let $a \in \operatorname{co} \bar{L}$. We shall prove that $a \in K$, i.e., that $a \in \partial \operatorname{co} L(t)$ for some $t \in [0, 1]$. Let

$$t = \inf \{\alpha : \alpha \geqslant 0, a \in \operatorname{co} \bar{L}(\alpha)\}.$$

If $a \notin \operatorname{co} \bar{L}(t)$ then there is an $\epsilon > 0$ such that

$$N(a, \epsilon) \cap \operatorname{co} \bar{L}(t) = \phi,$$

there is a $\delta > 0$ such that $| \int_D f d\mu | \leqslant \epsilon/2$ for all $D \in \mathscr{A}$ with $\mu(D) \leqslant \delta$ and then $a \notin \operatorname{co} \bar{L}(t + \delta)$ which cannot be by construction.

If $a \in \operatorname{int} \operatorname{co} \bar{L}(t)$ then there is an $\epsilon > 0$ such that $N(a, \epsilon) \subset \operatorname{co} \bar{L}(t)$, there is a $\delta > 0$ such that $| \int_D f d\mu | \leqslant \epsilon/2$ for all $D \in \mathscr{A}$ with $\mu(D) \leqslant \delta$ and then $a \in \operatorname{co} \bar{L}(t - \delta)$ which cannot be by construction.

We have then $a \in \partial \operatorname{co} \bar{L}(t) \subset \partial \operatorname{co} L(t)$. This concludes the proof of Lemma I.

LEMMA II.

$$\bigcup_{\alpha \in [0,1]} \partial \operatorname{co} L(\alpha) \subset L.$$

PROOF OF LEMMA II. It is enough to prove that for any $\alpha \in [0, 1]$ we have $\partial \operatorname{co} L(\alpha) \subset L(\alpha)$. To simplify the notations we shall give this proof only for $\alpha = 1$, i.e., we shall prove that $\partial \operatorname{co} L \subset L$. The proof for any $\alpha \in [0, 1)$ is similar. Let $U = \{p : p \in X, | p | = 1\}$. From now on the variable p will always be restricted to the set U. Since the function f is Lebesgue integrable and $\mathscr{A} \subset \mathscr{B}$ we know that the set L is bounded. For each $p \in U$ we define the following items:

$$m(p) = \sup_{x \in L} p \cdot x$$

$$H(p) = \{x : p \cdot x = m(p)\}$$

$$G(p) = L \bigcap H(p)$$

$$D^+(p) = \{t : p \cdot f(t) > 0, t \in [0, 1)\}$$

$$D^0(p) = \{t : p \cdot f(t) = 0, t \in [0, 1]\}.$$

We have then $D^+(p)$ and $D^0(p) \in \mathscr{A}$. Let $f_p \in \mathscr{A}$ be defined as follows:

$$\begin{aligned} f_p(t) &= f(t) \qquad \text{for} \qquad t \in D^0(p) \\ &= 0 \qquad \text{for} \qquad t \in [0, 1] \sim D^0(p). \end{aligned}$$

Let

$$G_0(p) = L(f_p, \mathscr{A})$$

It is a trivial matter to prove that

$$G(p) = \left\{ \int_{D^+(p)} f d\mu + \alpha : \alpha \in G_0(p) \right\}.$$

Let n be the dimension of the finite dimensional Euclidean space X. The proof of Lemma II will proceed by induction on n.

We consider first the case $n = 1$. We have then

$$\partial \operatorname{co} L = \{-m(-1), +m(+1)\},$$

$$\int_{D^+(+1)} f d\mu = m(+1) \quad \text{and} \quad \int_{D^+(-1)} f d\mu = -m(-1).$$

Since $D^+(+1)$ and $D^+(-1) \in \mathscr{A}$ it follows that $m(+1)$ and $-m(-1) \in L$. This concludes the proof of Lemma II in the case $n = 1$.

Let us assume that Lemma II is true for $n = k$ and prove it for $n = k + 1$. If Lemma II is true for $n = k$ it follows that Theorem I is also true for $n = k$.

We are now assuming that the Euclidean space X has dimension $k + 1$. For any $p \in U$ the set $G(p)$ is at most of dimension k and is closed and convex since the set $G^0(p)$ is closed and convex by Theorem I for $n = k$. Let $a \in \partial \operatorname{co} L$ then $a \in H(p)$ for some $p \in U$. We conclude by proving that $a \in G(p)$.

In order to prove that $a \in G(p)$ we shall show that $a \notin G(p)$ leads to a contradiction. If $a \notin G(p)$ let b be the element of $G(p)$ closest to a. Since $G(p)$ is closed and convex b exists and is unique and $|a - b| \neq 0$. For any $\epsilon \geqslant 0$ let $p_\epsilon = p + \epsilon(a - b)$ and

$$D_\epsilon = (D^+(p) \sim D^+(p_\epsilon)) \cup ((D^+(p_\epsilon) \cup D^0(p_\epsilon)) \sim (D^+(p) \cup D^0(p))).$$

We have immediately $\lim_{\epsilon\to 0} \mu(D_\epsilon) = 0$ and $\lim_{\epsilon \to 0} \int_{D_\epsilon} |f(t)|\, dt = 0$. Hence there exists an $\eta > 0$ such that $\int_{D_\epsilon} |f(t)|\, dt \leqslant |a - b|/2$ for all $\epsilon \leqslant \eta$. Let $a^* \in G(p_\eta)$. Then there is a set $A^* \in \mathscr{A}$ with

$$D^+(p_\eta) \subset A^* \subset D^+(p_\eta) \cup D^0(p_\eta)$$

such that $\int_{A^*} f\, d\mu = a^*$. Let $B^* = D^+(p) \cup (A^* \cap D^0(p))$ and $b^* = \int_{B^*} f\, d\mu$. We have then $b^* \in G(p)$. Moreover we have also

$$(B^* \sim A^*) \cup (A^* \sim B^*) \subset D_\eta$$

which implies $|a^* - b^*| \leqslant |a - b|/2$. By construction we have

$$(p + \eta(a - b)) \cdot a \leqslant (p + \eta(a - b)) \cdot a^*$$

and

$$p \cdot a^* \leqslant p \cdot a$$

hence

$$(a - b) \cdot (a^* - a) \geqslant 0\,.$$

Since the vector $a - b$ is an outward normal to a supporting hyperplane of the convex set $G(p)$ at the point b and since $b^* \in G(p)$ we have also

$$(a - b) \cdot (b - b^*) \geqslant 0$$

From the last two inequalities we obtain immediately

$$(a - b) \cdot (a^* - b^*) \geqslant (a - b) \cdot (a - b)$$

which implies

$$| a^* - b^* | \geqslant | a - b |$$

and contradicts the relation

$$| a^* - b^* | \leqslant | a - b |/2$$

obtained earlier. This contradiction concludes the proof of Lemma II.

Acknowledgments

The author is most grateful to Drs. Lucien W. Neustadt, Richard F. Datko, and Henry G. Hermes for their valuable comments on this paper.

References

1. D. Blackwell. The range of a vector measure. *Proc. Am. Math. Soc.* **2**, 390-395 (1951).
2. A. Dvoretzky, A. Wald, and J. Wolfowitz. Relations among certain ranges of vectors measures. *Pacific J. Math.* 59-74 (1951).
3. H. Halkin. "On the Necessary Condition for Optimal Control of Nonlinear Systems," Tech. Rept. No. 116 (Department of Mathematics, Stanford University, 1963). *J. Anal. Math.* **12**, 1-82 (1964).
4. P. R. Halmos. The range of a vector measure. *Bull. Am. Math. Soc.* **54**, 416-421 (1948).
5. S. Karlin. Extreme points of vector functions. *Proc. Am. Math. Soc.* **4**, 603-610 (1953).
6. A. Lyapounov. Sur les fonctions vecteurs complètement additives. *Bull. Acad. Sci. URSS* **4**, 465-478 (1940) (in Russian with a French résumé).
7. J. P. LaSalle. The time optimal control problem, *in* "Contributions to the Theory of Nonlinear Oscillations," Vol. V, pp. 1-24. Princeton Univ. Press, 1960.
8. H. Halkin. Lyapunov's theorem on the range of a vector measure and Pontryagin's maximum principle. *Arch. Rat. Mech. Anal.* **10**, 296-304 (1962).
9. L. W. Neustadt. The existence of optimal control in the absence of convexity conditions. *J. Math. Anal. Appl.* **7**, 110-117 (1963).
10. A. F. Filippov. On certain questions in the theory of optimal control. *Vestnik Moskov. Univ., Ser. Mat., Mekh., Astron., Fiz., Khim,* No. 2, 25-32 (1959). English translation in *J. Soc. Ind. Appl. Math., Ser. A*, 76-84 (1962).
11. J. Warga. Relaxed variational problems. *J. Math. Anal. Appl.* **4**, 111-128 (1962).